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A New Shooting Method for Multi-point Discrete Boundary Value Problems

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1. INTRODUCTION

Recently discrete boundary value problems have attracted many workers [9, 11–14] who have obtained several interesting theoretical results. Several constructive methods of solution have also been proposed, e.g., the method of complementary functions, the method of adjoints, the method of particular solutions [1–4], invariant imbedding methods [6, 13], etc. All these methods convert the linear boundary value problem to its equivalent initial value problem in just one pass, i.e., no iteration is needed. The purpose of this paper is to propose another one-pass practical shooting type method for multi-point boundary value problems. The origin of this method is the method of chasing developed by Gel'fand and Lokutsiyevskii for second-order linear differential equations [7, 16] and for higher-order equations [5]. The applicability of the proposed method is illustrated by solving several examples which are known to be numerically unstable.

2. LINEAR BOUNDARY VALUE PROBLEMS

Consider the linear difference equation

$$E^n u(t) = \sum_{i=0}^{n-2} b_i(t)\, E^{n-i-2} u(t) + f(t), \tag{1}$$

where t belongs to the discrete set $\mathcal{N} = \{0, 1, 2, \ldots\}$ and E is the shifting operator $Eu(t) = u(t+1)$. The functions $b_i(t)$, $0 \leqslant i \leqslant n-3$, $b_{n-2}(t) \neq 0$, and $f(t)$ are assumed to be defined for all $t \in \mathcal{N}$.

The boundary conditions are

$$\sum_{i=0}^{n-1} c_{ij} E^{n-i-1} u(t_j) + \alpha_j = 0, \qquad 1 \leqslant j \leqslant n, \tag{2}$$

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where $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ and $t_j \in \mathcal{N}$ for all $1 \leqslant j \leqslant n$. The coincidence of several points means that at t_i several conditions are prescribed, which are assumed to be linearly independent. Therefore, at t_j at least one of c_{ij}, $0 \leqslant i \leqslant n-1$, is not zero, say, $c_{(n-k-1)j} \neq 0$. Then, at t_j Eq. (2) can be rewritten as

$$E^k u(t_j) = \sum_{\substack{i=0 \\ i \neq n-k-1}}^{n-1} d_{ij} E^{n-i-1} u(t_j) + \beta_j, \qquad 0 \leqslant k \leqslant n-1, \tag{3}$$

where

$$d_{ij} = -c_{ij}/c_{(n-k-1)j}, \qquad 0 \leqslant i \leqslant n-1,\ i \neq n-k-1,$$

and

$$\beta_j = -\alpha_j/c_{(n-k-1)j}.$$

To begin with we create a linear difference equation of $(n-1)$th order based on the form of Eq. (3) containing n unknown functions. Such an equation can be written as

$$E^k u(t) = \sum_{\substack{i=0 \\ i \neq n-k-1}}^{n-1} p_{ij}(t)\, E^{n-i-1} u(t) + q_j(t), \tag{4}$$

where $t \in \mathcal{N}$. To find the unknown functions $p_{ij}(t)$, $0 \leqslant i \leqslant n-1$, $i \neq n-k-1$, and $q_j(t)$, we rewrite Eq. (4) as

$$E^{k+1} u(t) = \sum_{\substack{i=0 \\ i \neq n-k-1}}^{n-1} Ep_{ij}(t)\, E^{n-i} u(t) + Eq_j(t). \tag{5}$$

Next, we shall use (4) to eliminate $E^{n-1}u(t)$ from (5), however, it depends on a particular value of k and we need to consider four different cases:

(i) $k=0$ and $n \geqslant 3$:

$E^{n-1}u(t)$ as obtained from (4) is

$$E^{n-1} u(t) = \frac{1}{p_{0j}(t)} \left[u(t) - \sum_{i=1}^{n-2} p_{ij}(t)\, E^{n-i-1} u(t) - q_j(t) \right]. \tag{6}$$

Using (6) in (5), we obtain

$$\begin{aligned} E^n u(t) = {} & \frac{Eu(t)}{Ep_{0j}(t)} - \frac{Ep_{1j}(t)}{p_{0j}(t)\, Ep_{0j}(t)} \left[u(t) - \sum_{i=1}^{n-2} p_{ij}(t)\, E^{n-i-1} u(t) - q_j(t) \right] \\ & - \sum_{i=1}^{n-3} \frac{Ep_{(i+1)j}(t)}{Ep_{0j}(t)}\, E^{n-i-1} u(t) - \frac{Eq_j(t)}{Ep_{0j}(t)}. \end{aligned} \tag{7}$$

Comparing (1) with (7), we see that the following system of n difference equations must be satisfied:

$$\begin{aligned} Ep_{0j}(t) &= [b_{n-2}(t)\, p_{(n-2)j}(t) + b_{n-3}(t)]^{-1}, \\ Ep_{1j}(t) &= -b_{n-2}(t)\, p_{0j}(t)\, Ep_{0j}(t), \\ Ep_{(i+1)j}(t) &= -[b_{i-1}(t) + b_{n-2}(t)\, p_{ij}(t)]\, Ep_{0j}(t), \qquad 1 \leqslant i \leqslant n-3, \\ Eq_j(t) &= -[f(t) + b_{n-2}(t)\, q_j(t)]\, Ep_{0j}(t). \end{aligned} \tag{8}$$

We also desire that the solution of (4) must satisfy the boundary condition (3). For this, we compare (3) and (4) at the point t_j and find

$$\begin{aligned} p_{ij}(t_j) &= d_{ij}, \qquad 0 \leqslant i \leqslant n-2, \\ q_j(t_j) &= \beta_j. \end{aligned} \tag{9}$$

In the remaining three cases, we proceed as for the case $k=0$ and $n \geqslant 3$, and arrive at the following systems of difference equations:

(ii) $1 \leqslant k \leqslant n-3$:

$$\begin{aligned} Ep_{0j}(t) &= p_{(n-1)j}(t)[b_{n-k-3}(t)\, p_{(n-1)j}(t) - b_{n-2}(t)\, p_{(n-k-2)j}(t)]^{-1}, \\ Ep_{1j}(t) &= p_{0j}(t)\, b_{n-2}(t)\, Ep_{0j}(t)/p_{(n-1)j}(t), \\ Ep_{(i+1)j}(t) &= [Ep_{1j}(t)\, p_{ij}(t) - b_{i-1}(t)\, p_{0j}(t)\, Ep_{0j}(t)]/p_{0j}(t), \\ &\qquad 1 \leqslant i \leqslant n-2,\ i \neq n-k-1,\ n-k-2, \end{aligned} \tag{10}$$

$$\begin{aligned} Ep_{(n-k)j}(t) &= -[Ep_{1j}(t) + b_{n-k-2}(t)\, p_{0j}(t)\, Ep_{0j}(t)]/p_{0j}(t), \\ Eq_j(t) &= -[f(t)\, p_{0j}(t)\, Ep_{0j}(t) - Ep_{1j}(t)\, q_j(t)]/p_{0j}(t), \\ p_{ij}(t_j) &= d_{ij},\ 0 \leqslant i \leqslant n-1,\ i \neq n-k-1,\ q_j(t) = \beta_j. \end{aligned} \tag{11}$$

(iii) $k = n-2$:

$$\begin{aligned} Ep_{0j}(t) &= -p_{(n-1)j}(t)/(b_{n-2}(t)\, p_{0j}(t)), \\ Ep_{2j}(t) &= (1 - b_0(t)\, p_{0j}(t)\, Ep_{0j}(t))/p_{0j}(t), \\ Ep_{(i+1)j}(t) &= -(1 + b_{i-1}(t)\, p_{0j}(t)\, Ep_{0j}(t))/p_{0j}(t), \qquad 2 \leqslant i \leqslant n-2, \\ Eq_j(t) &= -(f(t)\, p_{0j}(t)\, Ep_{0j}(t) + q_j(t))/p_{0j}(t), \end{aligned} \tag{12}$$

$$\begin{aligned} p_{ij}(t_j) &= d_{ij}, \qquad 0 \leqslant i \leqslant n-1,\ i \neq 1, \\ q_j(t_j) &= \beta_j. \end{aligned} \tag{13}$$

(iv) $k=n-1$:

$$\begin{aligned} Ep_{1j}(t) &= b_{n-2}(t)/p_{(n-1)j}(t), \\ Ep_{(i+1)j}(t) &= b_{i-1}(t) - Ep_{1j}(t)\, p_{ij}(t), \qquad 1 \leqslant i \leqslant n-2, \\ Eq_j(t) &= f(t) - Ep_{1j}(t)\, q_j(t), \end{aligned} \tag{14}$$

$$\begin{aligned} p_{ij}(t_j) &= d_{ij}, \qquad 1 \leqslant i \leqslant n-1, \\ q_j(t_j) &= \beta_j. \end{aligned} \tag{15}$$

For the particular value of k, we solve the appropriate system from t_j to t_n and collect the values of $p_{ij}(t_n)$, $0 \leqslant i \leqslant n-1$, $i \neq n-k-1$, and $q_j(t_n)$, thereby obtaining from (4) a new boundary relations at t_n

$$E^k u(t_n) = \sum_{i=0}^{n-1} p_{ij}(t_n)\, E^{n-i-1} u(t_n) + q_j(t_n). \tag{16}$$

Let N be the number of different boundary points, i.e., $t_1 < t_2 < \cdots < t_N = t_n$ $(n \geqslant N \geqslant 2)$, and $m(t_j)$ represents the number of boundary relations (2) prescribed at the point t_j. Hence, $\sum_{j=1}^{N} m(t_j) = n$. Thus, in (2) we have $m(t_n)$ relations at the point t_n and to obtain $E^i u(t_n)$, $0 \leqslant i \leqslant n-1$, we need $n - m(t_n)$ more relations (16). This in turn implies that we need to solve $n - m(t_n)$ appropriate above systems. These systems are not necessarily different, especially because a difference system does not change as long as in (3) k is the same (we can have at most n different difference systems). Further, without loss of generality, we can assume that $m(t_n) = \max_{1 \leqslant j \leqslant N} m(t_j)$, otherwise the role of the point t_n with the point t_j where $m(t_j)$ is maximum can be interchanged. Finally, having obtained $E^i u(t_n)$, $0 \leqslant i \leqslant n-1$, we solve the difference equation (1) as an initial value problem from the right-hand end point t_n to the initial point t_1.

Obviously, the above procedure is meaningful only when the right sides of the appropriate system are defined. For example, consider the boundary value problem

$$\begin{aligned} E^2 u(t) &= b_0(t) u(t) + f(t), \\ u(0) = A, &\qquad u(T) = B. \end{aligned} \tag{17}$$

The relevent system for (17) can easily be obtained from the case (iii) and is given by

$$\begin{aligned} Ep_{01}(t) &= \frac{1}{p_{01}(t)\, b_0(t)} \\ Eq_1(t) &= -\left(q_1(t) + \frac{f(t)}{b_0(t)} \right) \frac{1}{p_{01}(t)} \end{aligned} \tag{18}$$

together with the initial conditions

$$p_{01}(0) = 0, \qquad q_1(0) = A.$$

Since $p_{01}(0) = 0$ the right sides of (18) are not meaningful. In such a situation we need to modify our representation (4). This is achieved by taking the form

$$E^k u(t_j) = \sum_{i=0}^{n-1} r_{ij}(t)\, E^{n-i-1} u(t) + s_j(t). \tag{19}$$

This implies the introduction of $(n+1)$ unknown functions $r_{ij}(t)$, $0 \leqslant i \leqslant n-1$, and $s_j(t)$.

Equation (19) leads to

$$E^k u(t_j) = \sum_{i=0}^{n-1} Er_{ij}(t)\, E^{n-i} u(t) + Es_j(t). \tag{20}$$

As earlier, the combination of (19), (20), and (1) leads to the system

$$\begin{aligned} Er_{0j}(t) &= r_{(n-1)j}(t)/b_{n-2}(t), \\ Er_{1j}(t) &= r_{0j}(t), \\ Er_{(i+1)j}(t) &= -Er_{0j}(t)\, b_{i-1}(t) + r_{ij}(t), \qquad 1 \leqslant i \leqslant n-2, \\ Es_j(t) &= -f(t)\, Er_{0j}(t) + s_j(t), \end{aligned} \tag{21}$$

$$\begin{aligned} r_{ij}(t_j) &= d_{ij}, \qquad 0 \leqslant i \leqslant n-1,\ i \neq n-k-1, \\ r_{(n-k-1)j}(t_j) &= 0, \\ s_j(t_j) &= \beta_j. \end{aligned} \tag{22}$$

Finally, this procedure can also be used to solve

$$E^n u(t) = h(t)\, E^{n-1} u(t) + \sum_{i=0}^{n-2} b_i(t)\, E^{n-i-2} u(t) + f(t), \qquad h(t) \not\equiv 0, \tag{23}$$

together with the boundary conditions (3). The combination of (19), (20), and (23) provides the same system as (21), (22) except in (21) the second equation is replaced by

$$Er_{1j}(t) = r_{0j}(t) - Er_{0j}(t)\, h(t). \tag{24}$$

3. Illustrative Examples

3.1. *Consider the definite integral*

$$u(t) = \int_0^1 x^t e^{x-1}\, dx, \qquad t = 1, 2, 3, \ldots . \tag{25}$$

It can easily be seen that $0 < u(t) < u(t-1)$ and $u(t) \to 0$ as $t \to \infty$, also

$$u(t+1) = 1 - (t+1)\, u(t). \tag{26}$$

With $u(1) = 1/e$ correct to any number of places, the recurrence relation (26) provides unrealistic values [10]. To overcome this difficulty, using the known behaviour of $u(t)$, Dorn and McCracken [10] took $u(N) = 0$ for sufficiently large N and recursed backward. To check the accuracy of results, they arbitrarily choose M ($> N$) and obtained another set of values of the integral. The search for M and N continues until the results agree to the desired degree of accuracy. However, this method does not appear to be practicable.

To evaluate the given integral we notice that $u(t)$ also satisfies

$$u(t+2) = (t+1)(t+2)\, u(t) - (t+1)$$

together with

$$u(1) = 1 - u(0), \tag{27}$$

and at sufficiently large N, (26) implies

$$u(N) = \frac{1}{N+2}.$$

To solve this following the procedure given in the preceeding section, we take

$$u(t+1) = p_{10}(t)\, u(t) + q_0(t) \tag{28}$$

and find (case (iv)) that the unknown functions $p_{10}(t)$ and $q_0(t)$ should satisfy

$$\begin{aligned} p_{10}(t+1) &= \frac{(t+1)(t+2)}{p_{10}(t)}, \\ q_0(t+1) &= -(t+1)\left[1 + \frac{t+2}{p_{10}(t)}\, q_0(t)\right], \\ p_{10}(0) &= -1, \qquad q_0(0) = 1. \end{aligned} \tag{29}$$

From this we obtain $p_{10}(N-1)$ and $q_0(N-1)$, and in turn using (28) compute $u(N-1)$. Once $u(N)$ and $u(N-1)$ are known, one can easily compute the solution from (27). Taking $N = 10{,}001$ we computed $u(t)$ and the results are presented in Table I.

TABLE I

Values of $\int_0^1 x^t e^{x-1}\,dx$

t		t		t		t	
1	0.36787944 $E+00$	10	0.83877070 $E-01$	100	0.98048550 $E-02$	1000	0.99800499 $E-03$
2	0.26424112 $E+00$	20	0.45544884 $E-01$	200	0.49506158 $E-02$	2000	0.49950062 $E-03$
3	0.20727665 $E+00$	30	0.31279674 $E-01$	300	0.33112945 $E-02$	3000	0.33311130 $E-03$
4	0.17089341 $E+00$	40	0.23822729 $E-01$	400	0.24875775 $E-02$	4000	0.24987508 $E-03$
5	0.14553294 $E+00$	50	0.19237754 $E-01$	500	0.19920398 $E-02$	5000	0.19992004 $E-03$
6	0.12680236 $E+00$	60	0.16133165 $E-01$	600	0.16611341 $E-02$	6000	0.16661113 $E-03$
7	0.11238350 $E+00$	70	0.13891533 $E-01$	700	0.14245043 $E-02$	7000	0.14281634 $E-03$
8	0.10093197 $E+00$	80	0.12196915 $E-01$	800	0.12468847 $E-02$	8000	0.12496876 $E-03$
9	0.91612293 $E-01$	90	0.10870836 $E-01$	900	0.11086488 $E-02$	9000	0.11108643 $E-03$

3.2. Dahlquist and Björck [8] considered

$$u(t)=\int_0^1 \frac{x^t}{5+x}\,dx, \qquad t=1,2,... \tag{30}$$

and found that the problem is numerically unstable. As in the last example, we see that $u(t)$ satisifes

$$\begin{aligned} u(t+2) &= 25u(t)-\frac{4t+9}{(t+1)(t+2)}, \\ u(1)+5u(0) &= 1, \end{aligned} \tag{31}$$

and since $u(\infty)=0$ and $u(t+1)+5u(t)=1/(t+1)$, we require

$$u(N)=\frac{1}{6(N+1)}.$$

To solve this we follow the similar procedure as in the preceeding example. Taking $N=10{,}001$ we obtained the solution and present the same in Table II.

3.3. In both preceding problems $u(\infty)=0$. We now consider

$$u(t)=\int_0^2 \frac{x^t}{5+x}\,dx, \qquad t=1,2,... . \tag{32}$$

For this, it is easy to see that $u(\infty)=\infty$. To obtain $u(t)$ we notice that

$$\begin{aligned} u(t+2) &= 25u(t)-\frac{(3t+8)\,2^{t+1}}{(t+1)(t+2)}, \\ u(1)+5u(0) &= \frac{2^{t+1}}{t+1}, \end{aligned} \tag{33}$$

TABLE II

Values of $\int_0^1 (x^t/(5+x)\,dx$

t		t		t		t	
1	0.88392216 $E-01$	10	0.15367550 $E-01$	100	0.16528701 $E-02$	1000	0.16652787 $E-03$
2	0.58038920 $E-01$	20	0.79975230 $E-02$	200	0.82987267 $E-03$	2000	0.83298623 $E-04$
3	0.43138734 $E-01$	30	0.54046330 $E-02$	300	0.55401577 $E-03$	3000	0.55540127 $E-04$
4	0.34306330 $E-01$	40	0.40812983 $E-02$	400	0.41580006 $E-03$	4000	0.41657988 $E-04$
5	0.28468352 $E-01$	50	0.32785146 $E-02$	500	0.33277852 $E-03$	5000	0.33327779 $E-04$
6	0.24324906 $E-01$	60	0.27396243 $E-02$	600	0.27739240 $E-03$	6000	0.27773920 $E-04$
7	0.21232615 $E-01$	70	0.23528767 $E-02$	700	0.23781206 $E-03$	7000	0.23806690 $E-04$
8	0.18836924 $E-01$	80	0.20618122 $E-02$	800	0.20811650 $E-03$	8000	0.20831163 $E-04$
9	0.16926490 $E-01$	90	0.18348317 $E-02$	900	0.18501384 $E-03$	9000	0.18516804 $E-04$

and since $u(t+1)+5u(t)=2^{t+1}/(t+1)$ we get

$$u(N)=\frac{2^{N+1}}{6(N+1)}.$$

To obtain the solution by following the procedure of Section 3.1, in view of limited computer capabilities we choose $N=250$ and present the values of $\int_0^2 (x^t/(5+x))\,dx$ in Table III.

3.4. Holt's two-point boundary value problem

$$\begin{aligned} \frac{d^2u}{dx^2} &= (2m+1+t^2)u, \\ u(0)=\beta, &\qquad u(\infty)=0, \end{aligned} \tag{34}$$

TABLE III

Values of $\int_0^2 (x^t/(5+x))\,dx$

t		t		t		t	
0	0.33647224 $E+00$	10	0.27260357 $E+02$	110	0.33498134 $E+31$	210	0.22311824 $E+61$
1	0.31763882 $E+00$	20	0.14456393 $E+05$	120	0.31460579 $E+34$	220	0.21812163 $E+64$
2	0.41180592 $E+00$	30	0.99861672 $E+07$	130	0.29751110 $E+37$	230	0.21367554 $E+67$
3	0.60763709 $E+00$	40	0.77149316 $E+10$	140	0.28300147 $E+40$	240	0.20971754 $E+70$
4	0.96181456 $E+00$	50	0.63425992 $E+13$	150	0.27056586 $E+43$	250	0.24027412 $E+73$
5	0.15909272 $E+01$	60	0.54252148 $E+16$	160	0.25982047 $E+46$		
6	0.27120305 $E+01$	70	0.47698737 $E+19$	170	0.25047150 $E+49$		
7	0.47255616 $E+01$	80	0.42792501 $E+22$	180	0.24229027 $E+52$		
8	0.83721921 $E+01$	90	0.38989275 $E+25$	190	0.23509613 $E+55$		
9	0.15027929 $E+02$	100	0.35960999 $E+28$	200	0.22874447 $E+58$		

TABLE IV

Solution of Holt's Problem for $m = 0$ and $\beta = 1$

x	Present solution	Complementary functions [19]	Finite difference [15]	Solution-by Osborne [17]	Roberts and Shipman [18]
0	0.10000000 E 01	0.10000000 E 01	0.100000 E 01	0.1000 E 01	0.10000000 E 01
1	0.25934256 E 00	0.15729920 E 00	0.157300 E 00	0.2593 E 00	0.15729921 E 00
2	0.34564050 $E-01$	0.46777349 $E-02$	0.467778 $E-02$	0.3455 $E-01$	0.46777350 $E-02$
3	0.19885236 $E-02$	0.22090497 $E-04$	0.220908 $E-04$	0.1987 $E-02$	0.22090497 $E-04$
4	0.45958211 $E-04$	0.15417257 $E-07$	0.154175 $E-07$	0.4590 $E-04$	0.15417259 $E-07$
5	0.41255782 $E-06$	0.15366706 $E-11$	0.153749 $E-11$	0.4188 $E-06$	0.15374602 $E-11$
6	0.14129838 $E-08$	-0.73163560 $E-15$	0.215201 $E-16$	0.1409 $E-08$	0.21519753 $E-16$
7	0.18272024 $E-11$	-0.75311525 $E-15$	0.418390 $E-22$	0.1821 $E-11$	0.41838334 $E-22$
8	0.88629432 $E-15$	-0.75315520 $E-15$	0.112244 $E-28$	0.8825 $E-15$	0.11224343 $E-28$
9	0.16054807 $E-18$		0.413703 $E-36$	0.1597 $E-18$	0.41370659 $E-36$
10	0.10827683 $E-22$		0.208844 $E-44$	0.1058 $E-22$	0.20895932 $E-44$
11	0.27127027 $E-27$		0.144078 $E-53$		0.12279100 $E-49$
12	0.25204672 $E-32$		0.135609 $E-63$		0.13487374 $E-49$
13	0.86739223 $E-38$				0.17299316 $E-60$
14	0.11045054 $E-43$				-0.25496486 $E-65$
15	0.51997977 $E-50$				
16	0.90444830 $E-57$				
17	0.58092510 $E-64$				
18	0.13771859 $E-71$				

where m and β are given constants, is an example where usual shooting methods do not yield satisfactory results. Various authors [15, 17–19] have therefore solved this problem by using different techniques. Its finite difference approximation [19] is given by

$$\begin{gathered} c_2(t)\, u(t+2) - c_1(t)\, u(t+1) + c_0(t)\, u(t) = 0, \\ u(0) = \beta, \qquad u(\infty) = 0, \end{gathered} \tag{35}$$

where

$$\begin{aligned} c_0(t) &= 1 - \tfrac{1}{12}h^2(2m + 1 + h^2t^2), \\ c_1(t) &= 2 + \tfrac{5}{6}h^2(2m + 1 + h^2(t+1)^2), \\ c_2(t) &= 1 - \tfrac{1}{12}h^2(2m + 1 + h^2(t+2)^2). \end{aligned}$$

To solve this problem, following the remarks after Eq. (23), we take

$$u(0) = r_{10}(t)\, u(t) + r_{00}(t)\, u(t+1) + s_0(t), \tag{36}$$

where unknown functions $r_{10}(t)$, $r_{00}(t)$, and $s_0(t)$ should satisfy

$$\begin{aligned} r_{10}(t+1) &= r_{00}(t) + \frac{c_1(t)}{c_0(t)}\, r_{10}(t), \\ r_{00}(t+1) &= -\frac{c_2(t)}{c_0(t)}\, r_{10}(t), \\ s_0(t+1) &= s_0(t), \\ r_{10}(0) = 1, \qquad & r_{00}(0) = 0, \qquad s_0(0) = 0. \end{aligned} \tag{37}$$

For given m, β, and h we compute $r_{10}(T)$, $r_{00}(T)$, and $s_0(T)$. These values in conjunction with $u(T+1) = 0$ enable us to obtain $u(T)$ from (36). Once $u(T+1)$ and $u(T)$ are known one can obtain the solution using the recurrence relation (35). To illustrate we have chosen $m = 0$, $\beta = 1$ with $h = 0.02$. Due to limited computer capabilities we have to take $T = 922$. Taking $u(923) = 0$, Eq. (36) can be used to compute $u(922)$. Having $u(923)$ and $u(922)$ one can easily compute the solution, which has been tabulated in Table IV, wherein we have also included the solution obtained by others. For other choices of m and β solution can be obtained in a similar manner.

References

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